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ARPA ORDER NO. 189-61

MEMORANDUM
RM-3848-ARPA
JULY 1964

PREDICTION OF SLBM IN-PLANE MOTION UNDER VARIOUS DEGREES OF A PRIORI KNOWLEDGE

F. B. Tuteur

PREPARED FOR:
ADVANCED RESEARCH PROJECTS AGENCY

The RAND Corporation
SANTA MONICA • CALIFORNIA

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PREFACE

This Memorandum is a product of a study for the Advanced Research Projects Agency of Defense Against Submarine-Launched Ballistic Missiles. One portion of this study is concerned with an airplane-based boost-intercept anti-missile missile system. This Memorandum treats the problem of observation and accurate prediction of the in-plane motion of an SLBM during the boost-phase portion of its trajectory, needed for efficient guidance of the interceptor. It analyzes the prediction accuracy which results when several different degrees of a priori knowledge about the boost trajectory are assumed. This Memorandum is a companion piece to an earlier Memorandum, RM-3606-ARPA, Early Estimation of SLBM Heading for Boost-Phase Interception (U), which considered the problem of heading estimation accuracy.

This report should be of interest to agencies and contractors concerned with the interception of ballistic missiles during their boost phase.

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SUMMARY

Observations on the trajectory of an SLBM can be used to estimate parameters of the trajectory and to predict future positions of the SLBM by using standard maximum-likelihood estimation methods. The prediction error is a function of the observation time and the length of time for which the prediction is made, and it also depends on the amount and kind of a priori information available. The prediction error is derived and computed for four different cases as follows:

1. The trajectory is a polynomial in time.
2. The trajectory corresponds to constant-thrust propulsion with unknown thrust parameters.
3. Constant-thrust trajectory with thrust parameters known but time and space origin unknown.
4. Constant-thrust trajectory with only time origin unknown.

In general, the greater the a priori knowledge the greater is the prediction accuracy.

ACKNOWLEDGMENTS

The author wishes to acknowledge the helpful discussions and suggestions of I. S. Blumenthal relevant to the delineation and presentation of the problems analyzed in this Memorandum. Appreciation is also due to D. G. Edelen and W. B. Kendall for their constructive reviews.

CONTENTS

LIST OF FIGURES

SYMBOLS

g = acceleration of gravity - ft/sec^2

I = specific impulse of rocket fuel - seconds

$\underline{J}$ = covariance matrix of estimated parameters

$\underline{N}$ = covariance matrix of noise

$p(s/R)$ = conditional probability density of s given R

r = rate of fuel expenditure - lbs/sec

$R(t)$ = ground range in ft

$R_c = R(t_c)$ = ground range at time $t = t_c$

$\delta R_c = \delta R(t_c)$ = range error at time $t = t_c$

s = unknown time origin of rocket trajectory

s_o = true time origin of rocket trajectory

t_1 = time observations cease and prediction is made

t_c = SLBM booster cut-off time

$u = \alpha_4 t_1$

$u_c = \alpha_4 t_c$

$\underline{X}$ = matrix of partial derivatives of range vs the parameters

W_o = weight of SLBM at launch - lbs

α_i = ith parameter

σ = standard deviation of individual observation error

τ = time interval between observations - time between radar pulses

I. INTRODUCTION

One of the major problems arising in the optimal guidance of an interceptor designed to intercept an SLBM during its boost-phase is the estimation of the parameters of the trajectory of the SLBM and the prediction of its location at any future time.[1] The parameters can be estimated from observations that have been made on the trajectory, but in general these observations are not accurate because of measurement noise. Some procedure is therefore needed to process the data in an optimum fashion. The most widely used method for doing this is the maximum likelihood method, which has been shown to be optimum from several points of view.[2,3] Therefore, in the following analysis this method is used.

The accuracy of the trajectory prediction is obviously a function of the length of time that the SLBM trajectory has been under observation as well as the amount of time into the future for which the prediction is made. In addition to these functional dependencies, the accuracy also depends very much on the amount and kind of a priori information available to the data processor. It is this particular dependence that is considered in this Memorandum. A number of different situations ranging from considerable a priori ignorance to almost complete a priori knowledge are analyzed.

That portion of the trajectory prediction problem which is concerned with estimating the heading or azimuth of the trajectory has been treated previously.[4] Since the trajectory is generally

nearly a straight line in the latter portions of the boost-phase, the problem of estimating elevation is similar to estimating the heading.

In this Memorandum attention is focused on predicting the SLBM position along its trajectory line. To simplify the discussions, the projection of that motion on the ground -- the ground range -- will be the quantity being predicted.

II. PROBLEM STATEMENT AND ASSUMPTIONS

The problem considered in this Memorandum is the prediction of the future ground range of an SLBM whose trajectory is assumed to lie in a vertical plane. It is supposed that the range has a known functional form $R(t)$, and that data are acquired and parameters are estimated continuously from the time $t = 0$ when the missile is first detected. The value of $R(t)$ is predicted for a future time t_c. The question is: What is the error in $R(t_c)$ resulting from a prediction made on data acquired and processed up to a time t_1 where $0 < t_1 < t_c$?

Although the time t_c can have any value, it will be thought of as the minimum cut-off time of the SLBM booster. This might also be the intercept time under the following conditions:(5)

a) The SLBM is on a least-favorable trajectory from the point of view of the interceptor.

b) Intercept is to take place during the boost phase of the SLBM trajectory.

c) The interceptor guidance policy minimizes the total fuel requirement.

In the examples, the following assumptions are made:

1. The only observed quantity considered by the estimator is $R(t)$, the ground range of the missile from the point of first observation. This quantity is computed by range and angle measurements made by the observing radar.

2. The observations consist of samples made at short, fixed time intervals. This assumption is made primarily to simplify the mathematics. However, it also corresponds quite closely to the operation of typical pulse radars.

3. The error in computed ground range due to measurement error at each observation instant has a gaussian amplitude distribution with zero mean and fixed variance. Actually for typical radars the variance increases with distance from the radar, but if the difference between initial and final distance is relatively small, the change in the variance is also small.

4. The measurement errors from sample to sample are uncorrelated. This assumption is again made for simplicity. If the measurement errors were assumed to be correlated, a form of the correlation would have to be specified. Also if the bandwidth of the radar receiver is approximately matched to the width of the radar pulses, there will be very little correlation between the samples.

III. RESULTS

The following four cases are considered:

1. R(t) can be approximated by a polynomial in t in which the coefficients multiplying powers of t are unknown, but where the time origin is known.
2. R(t) corresponds to a constant-thrust trajectory in which all parameters are unknown. A subcase, in which one of the unknown parameters (the initial velocity) is assumed to be known, is also considered under this heading to simplify the computations and to permit a check with results obtained with the digital computer.
3. R(t) corresponds to a constant-thrust trajectory in which only the initial value and the time origin are unknown.
4. R(t) corresponds to a constant-thrust trajectory in which only the time origin is unknown.

CASE 1. R(t) IS KNOWN TO BE A CUBIC POLYNOMIAL IN t

Here it is assumed that the range function R(t) can be approximated by a cubic polynomial in t during the time interval $0 < t < t_c$. Thus

$$R(t) = \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3 \tag{1}$$

where α's are unknown parameters. Typical trajectories, such as the one assumed in Appendix E (Fig. 8) to provide the common basis for all the numerical results of all the analyses in this Memorandum, can be approximated very closely by a quadratic

polynomial for $t < 25$ sec, and a cubic approximation is therefore good for most of the time interval of interest.

The computations for the prediction error are straightforward and are given in Appendix A. The rms prediction error is given by:

$$\delta R(t_e)_{rms} = \left\{ \frac{15\sigma^2 \tau t_c^{\,2}}{t_1^{\,3}} \left[20 - 120\frac{t_c}{t_1} + 276\left(\frac{t_c}{t_1}\right)^2 - 280\left(\frac{t_c}{t_1}\right)^3 + 105\left(\frac{t_c}{t_1}\right)^4 \right] \right\}^{1/2} \tag{2}$$

A graph of this expression for $t_e = 50$ sec is shown in Fig. 1. Note that since $R(t)$ is a linear function of the α's, the result is independent of the α's (see discussion following Eq. (26) of Appendix B).

CASE 2. SLBM MOTION IS KNOWN TO RESULT FROM CONSTANT-THRUST PROPULSION

The constant-thrust type of trajectory is more typical of the motion of SLBM trajectories during the boost phase than the polynomial expression considered in Case 1. Here the acceleration is given by:

$$\ddot{R}(t) = \frac{gIr}{W_o - rt} \tag{3}$$

where g = acceleration of gravity in ft/sec^2

I = specific impulse of the rocket fuel in sec

W_o = initial weight of the SLBM in lb

r = rate of fuel expenditure in lb/sec

This equation is integrated twice to yield

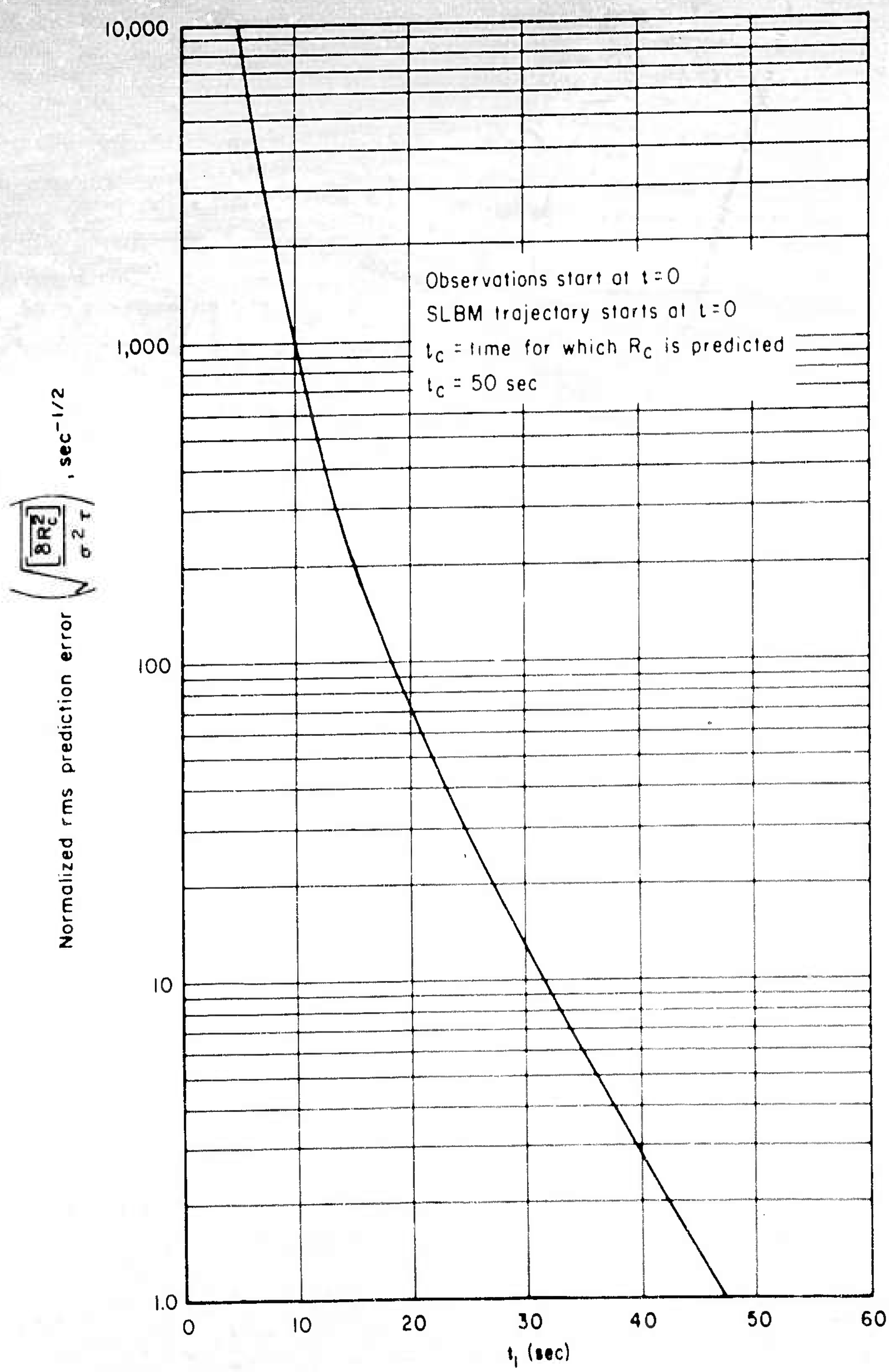

Fig. 1 — General normalized rms prediction error when R(t) is known to be a cubic polynomial

$$R(t) = \alpha_1 + \alpha_2 t + \frac{\alpha_3}{\alpha_4}\left[\left(1 - \alpha_4 t\right)\log\left(1 - \alpha_4 t\right) + \alpha_4 t\right] \qquad (4)$$

where $\alpha_1 = R(0)$, the range at $t = 0$ in ft

$\alpha_2 = \dot{R}(0)$, the velocity at $t = 0$ in ft/sec

$\alpha_3 = gI$, ft/sec

$\alpha_4 = r/W_o$, sec^{-1}

$t = 0$ corresponds to time of first observation.

In this case the α's are all assumed to be unknown. The maximum likelihood estimation of the parameter and the computation for the predicted error are now somewhat more involved than in Case 1, since $R(t)$ is no longer linear in the α's. Specifically, the error now depends on the assumed true value of α_4, since $R(t)$ is nonlinear in this parameter. (See discussion following Eq. (26) of Appendix B.) The details of the computations are contained in Appendix B where two subcases have been considered. In the first case it was assumed that the parameter $\alpha_2 = 0$.* This reduces the problem to an estimation of three parameters and permits an approximate analytical solution to be obtained, which holds for small values of t_1.

For large t_1 and for the complete four-parameter problem the computations were performed by a digital computer. The composite results are shown in Fig. 2 for $\alpha_4 = .0143\ \text{sec}^{-1}$ and $t_c = 50$ sec-- which are the parameters of the typical trajectory used in this Memorandum (Fig. 8 of Appendix E). The curve shows a somewhat larger prediction error than

*Corresponds to zero (or known) initial velocity.

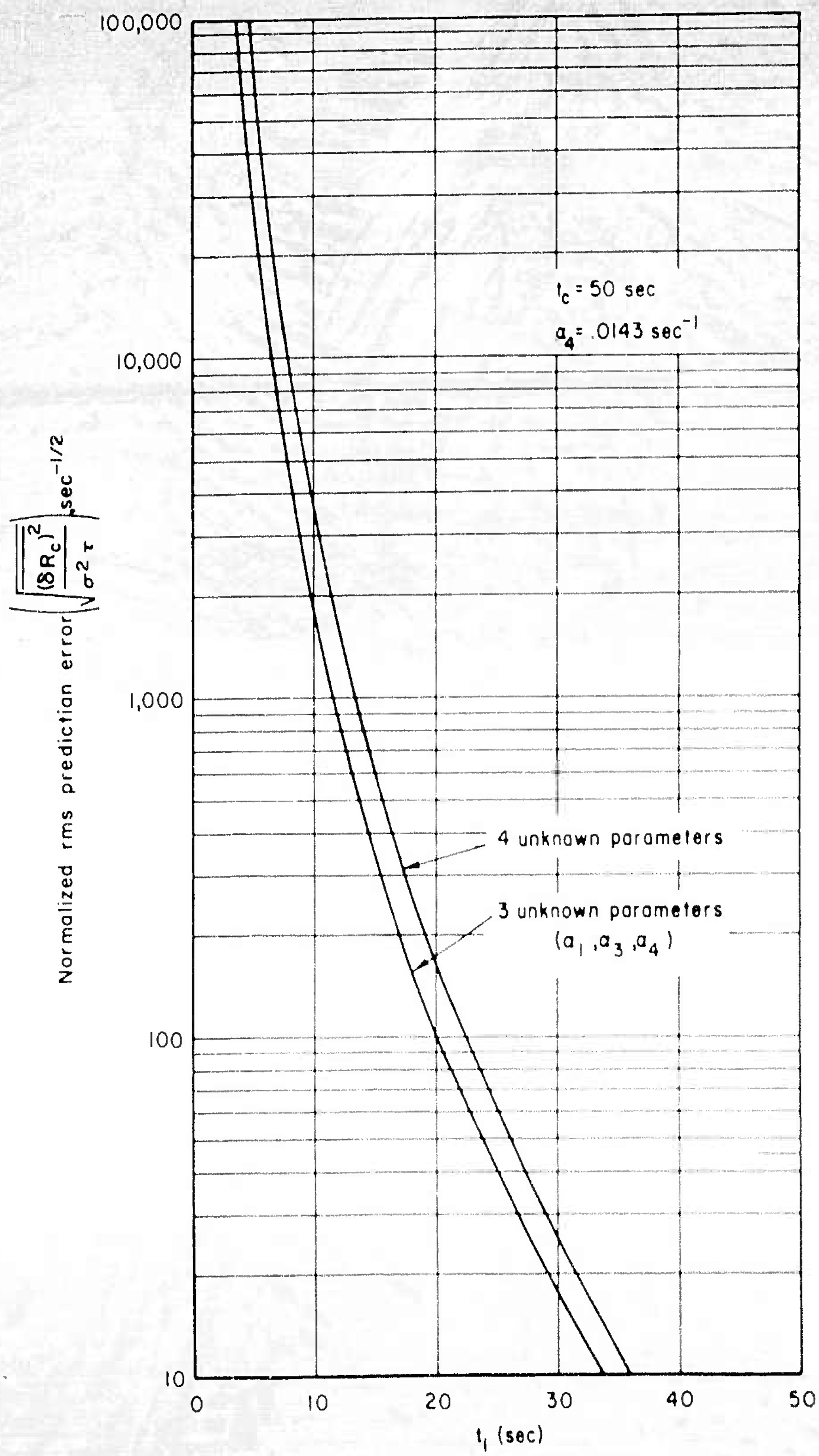

Fig. 2 — Normalized rms prediction error for motion known to be constant-thrust

the previous case, apparently because the constant-thrust trajectory is initially not very sensitive to changes in some of the parameters. This appears to result in inaccurate estimates of these parameters.

CASE 3. SLBM MOTION IS KNOWN TO RESULT FROM CONSTANT-THRUST PROPULSION AND ALL PARAMETERS EXCEPT TIME ORIGIN AND STARTING POINT ARE KNOWN

The relatively large prediction error observed in Case 2 arises from the somewhat unrealistic assumption that nothing whatever is known about the parameters of the range function. In practice, it is quite possible that most of the parameters are quite well known, as a result of technical intelligence, or otherwise. This applies especially to the parameters α_3 and α_4 which can be assumed to be exactly known if the type of rocket employed in the SLBM booster is known. On the other hand, it has been assumed in the previous two cases that the time origin of the range function is known to the observer. In general this will not be true, since the missile is not usually detected immediately upon its emergence. Thus, in the present case, it is assumed that the range function is given by

$$R(t) = \alpha_1 + \alpha_2(t-s) + \frac{\alpha_3}{\alpha_4}\left\{\left[1-\alpha_4(t-s)\right]\log[1-\alpha_4(t-s)] + \alpha_4(t-s)\right\} \qquad (5)$$

where the parameters α_2, α_3, and α_4 are assumed to be known exactly and the parameters α_1 and s are assumed to be unknown. In order to consider the possibility that some information about α_1 (the range from the point at which the SLBM emerges) might be available from

an independent radar observation, it is assumed that an a priori probability distribution is known for α_1. The computations of this case are presented in Appendix D, where it is shown that the a priori probability distribution for α_1 has very little effect on the prediction error. The results for a typical set of parameters are given in Fig. 3, and they show that the prediction error is considerably smaller than for Case 2, as would be expected.

CASE 4. SLBM MOTION IS KNOWN TO BE OF THE CONSTANT-THRUST TYPE WITH ONLY THE TIME ORIGIN BEING UNKNOWN

This case is quite similar to the previous one except that α_1 is now also assumed to be known. The computations for this case are given in Appendix C, and a graph of the results is shown in Fig. 4. The rms prediction error is, of course, somewhat smaller than for Case 3.

A comparison of all four cases is given in Fig. 5. Cases 3 and 4 are represented by only a single curve, that for which the actual time origin is zero; i.e., it coincides with the time origin for the observations.

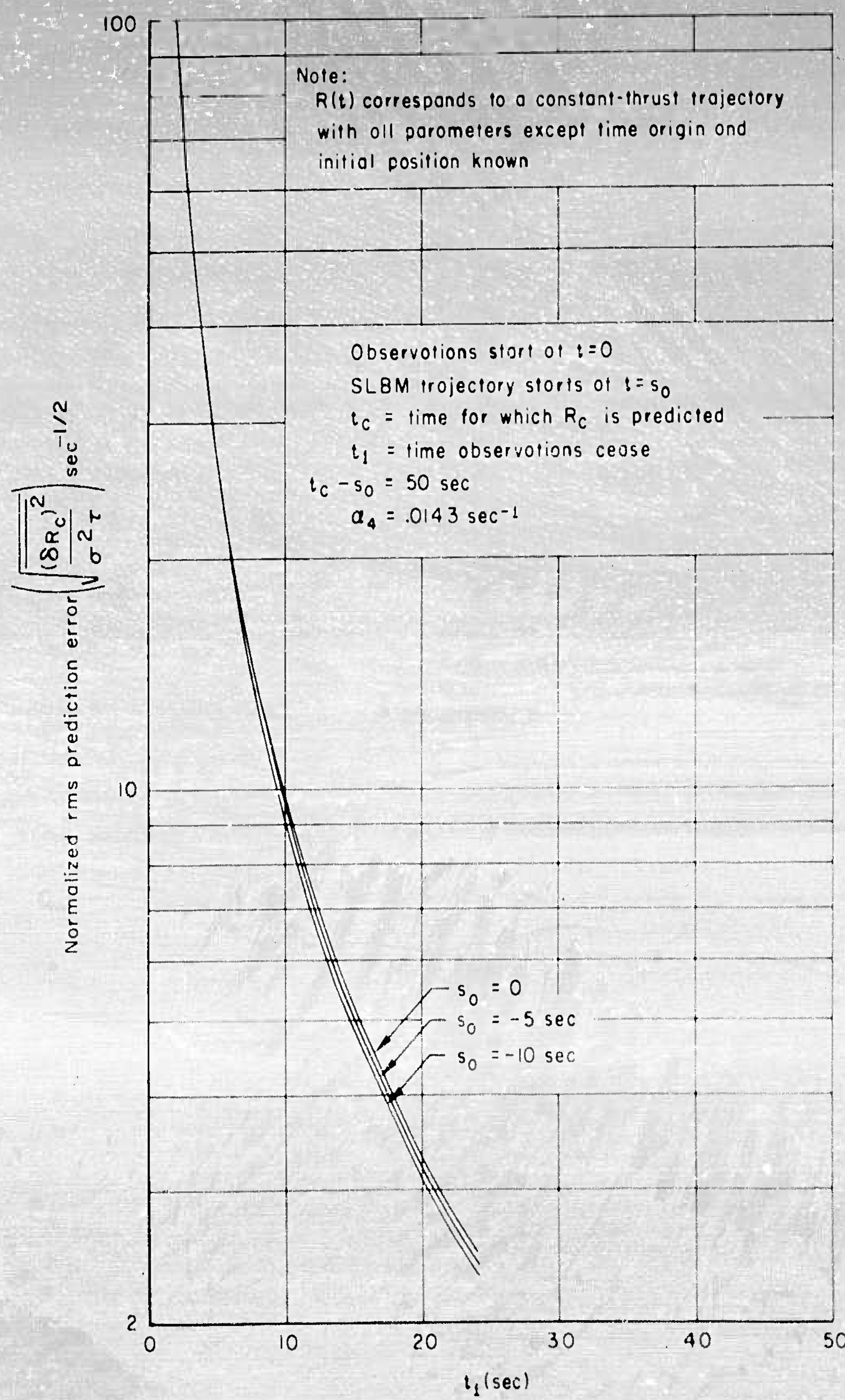

Fig. 3 — Normalized rms prediction error

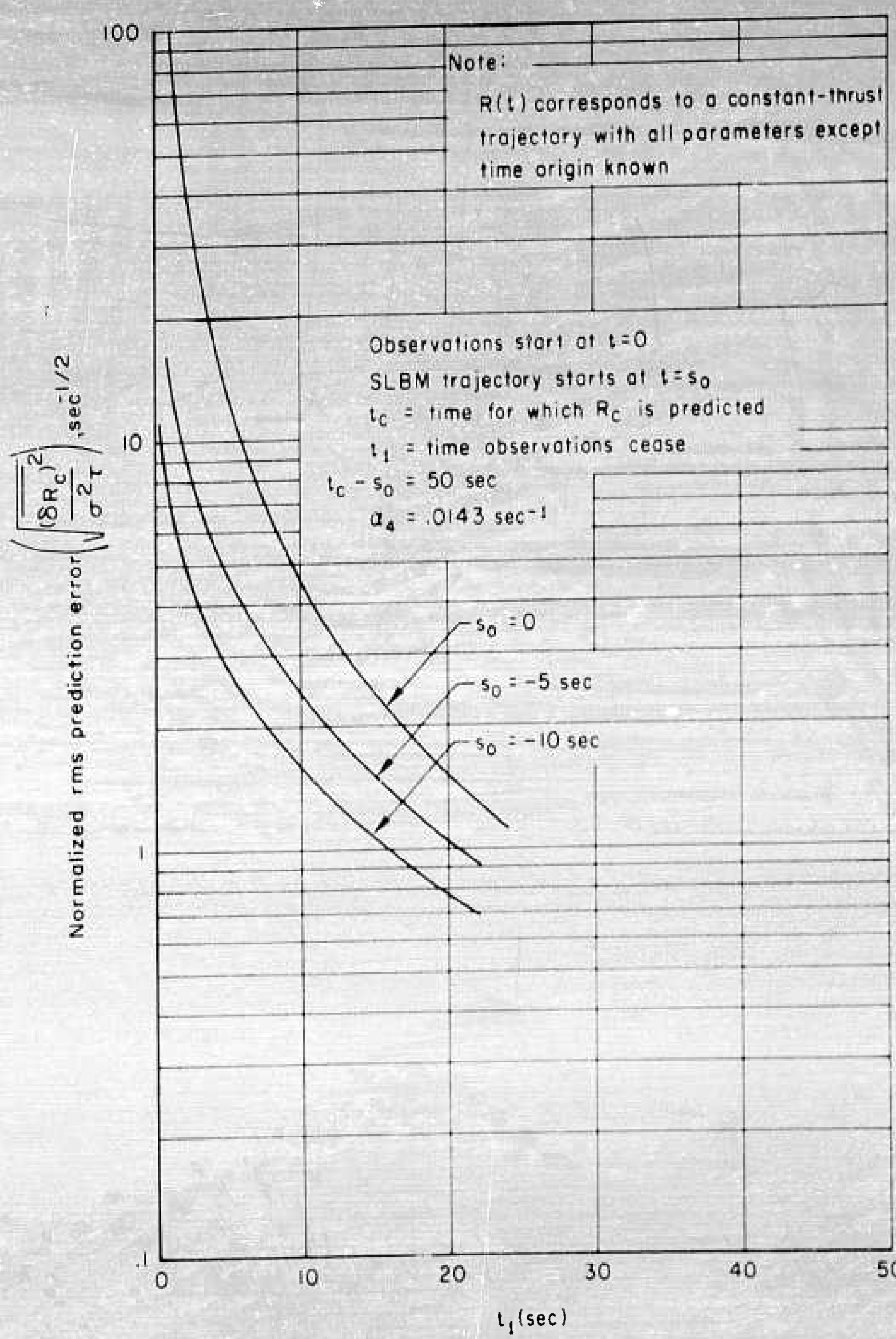

Fig. 4 — Normalized rms prediction error

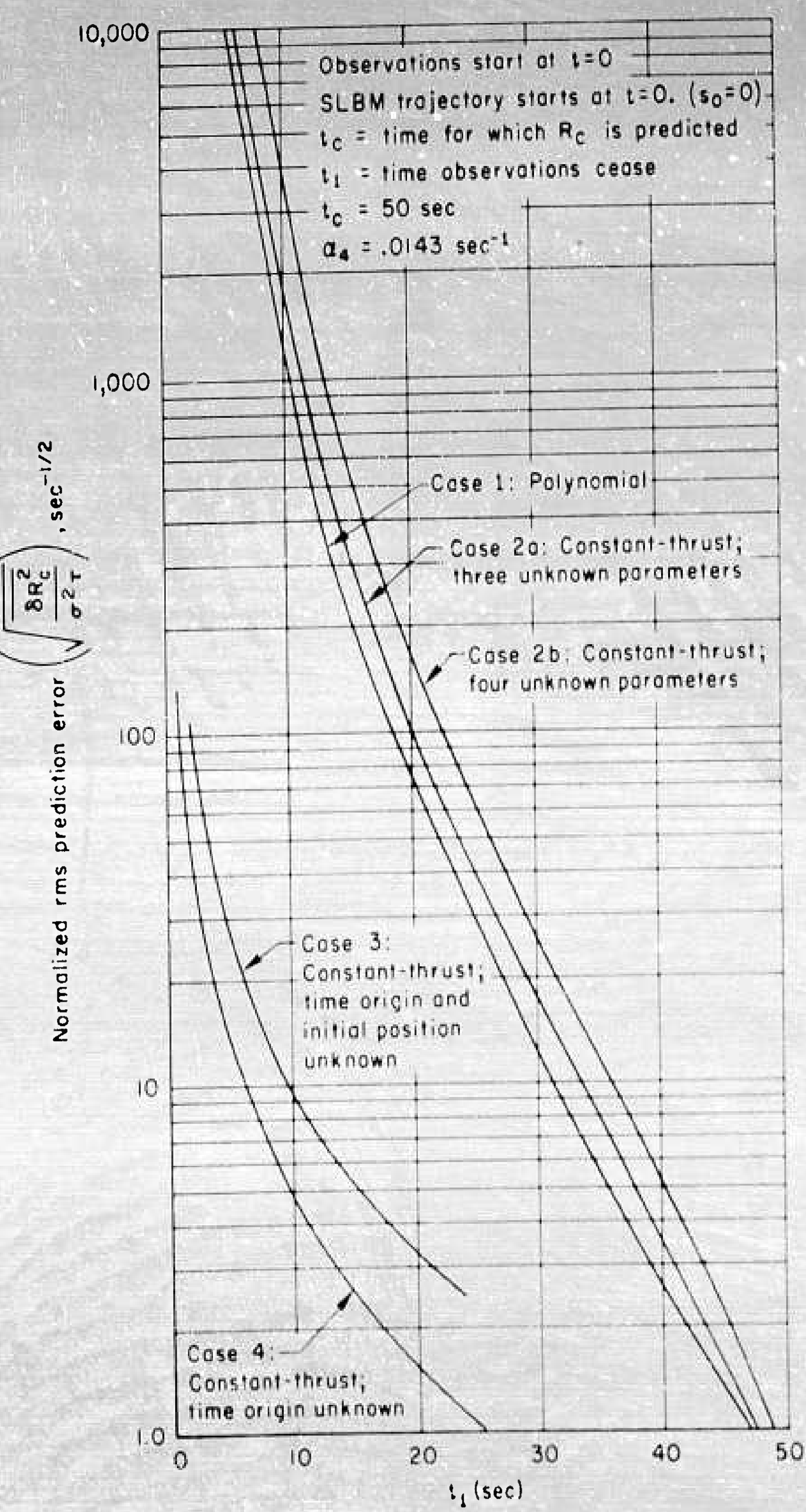

Fig. 5 — Normalized rms prediction error for five different states of a priori knowledge

IV. APPROXIMATE EFFECT OF A-PRIORI LIMITS ON THE PREDICTION ERROR

The computations of the prediction error made in Section III ignore any a priori information that might be available, and they result in an unreasonably large prediction error for small t_1. The a priori information can be taken into account very roughly by assuming that the prediction error is given by the maximum-likelihood computations of Section III whenever this is less than the a priori maximum error, and that the error is otherwise equal to the a priori maximum. The error in prediction then behaves roughly as shown in Fig. 6, and shows a pronounced threshold. The time of this threshold is a function of the a priori maximum error, and of the parameters affecting the maximum likelihood estimate. If, for example, one assumes that the prediction error cannot exceed 50,000 ft, and if σ is 5000 ft, and τ = .01 sec, then the threshold occurs approximately at the following times for the four cases considered:

Case	Threshold Time
1	18.2 sec
2a	19.8 sec
2b	22.4 sec
3	1.8 sec
4	1 sec

For values of t_1 greater than the threshold value, the error decreases very rapidly in all cases.

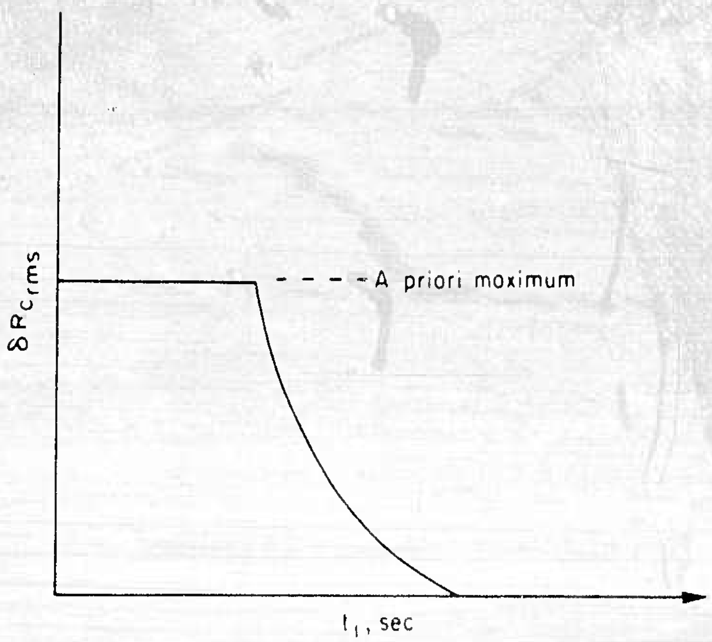

Fig. 6 — Range uncertainty versus time

V. CONCLUSIONS

The four examples of prediction of SLBM trajectories illustrate the way in which a priori information about the trajectory affects the prediction error. If the trajectory is in the form of a polynomial, which means that all time derivatives above a certain order are zero, then standard estimation theory is easily applied and the prediction error decreases very rapidly with the length of observation time. In practice, SLBM trajectories are often of the constant-thrust type. This means that their functional form is known, and one would therefore expect that the prediction error should be about the same as for the case where the trajectory is in the form of a polynomial. This is approximately true as shown by the results of cases 2a and 2b; however, the error in the constant-thrust case is uniformly higher than for the polynomial. The increased error is probably caused by the fact that the range function is initially not very sensitive to one of the parameters. When it is assumed that the rocket parameters are known, as in Cases 3 and 4, very small prediction errors are obtained.

It is clear that the results obtained in this Memorandum are based on the assumption that the trajectories depend on a small set of unknown but fixed parameters, and that the functional dependence is known. If actually a different functional dependence from the one assumed exists, or if important parameters are omitted from the estimation and prediction process, the errors will obviously be much greater than those obtained here.

This observation applies particularly to the problem of predicting the intercept point when staging of the SLBM is considered. The error in this case depends heavily, of course, on the amount of a priori information available.

In the most favorable case (from the point of view of the interceptor), the parameters of the two-stage trajectory are completely known, including the staging time. In this case the two-stage problem is essentially similar to Cases 3 and 4, considered in Section III, and accurate predictions can be made after only a short observation time. If only the staging time is unknown, however, observations made prior to staging cannot result in a prediction error that is smaller than the a priori range uncertainty corresponding to the a priori uncertainty in staging times.

In the most general and most difficult case, when little is known about the trajectory parameters, the problem raised by staging is that of estimating a set of parameters subject to an arbitrary change at an arbitrary time. So far as the author knows, this problem has not been completely solved at the present time.[6,7]

Appendix A

PREDICTION OF SLBM IN-PLANE MOTION WHEN R(t) IS KNOWN TO BE A THIRD-ORDER POLYNOMIAL

Suppose that R(t) can be approximated during the time interval $0 < t < t_c$ by a polynomial expansion of the form:

$$R(t) = \alpha_1 t + \alpha_2 t^2 + \alpha_3 t^3 \tag{6}$$

There is no difficulty in principle in considering higher-order polynomials, except that the mathematics become more unwieldy. The qualitative results should not be affected by the order of the approximation in any case. Note that Eq. (6) implies that the SLBM is launched at the time $t = 0$, and that the location of the launch point and the time of launch are known.

We assume that the parameters α_1, α_2, and α_3 are to be estimated from data taken on R(t) only. The acquisition of data and the estimation of the parameters is assumed to take place continuously from the time the target is launched. Then the question is: What is the error in $R(t_c)$ resulting from a prediction made on data acquired and processed up to a time t_1 where $0 < t_1 < t_c$?

For a third-order fit, we have from Eq. (6) that the error in $R(t_c)$ is

$$\delta R(t_c) = \delta\alpha_1 t_c + \delta\alpha_2 t_c^2 + \delta\alpha_3 t_c^3 \tag{7}$$

where $\delta\alpha_i$ is the error in estimating α_i. The uncertainty in $R(t_c)$ is best described by the variance given by:

$$\overline{\left[\delta R(t_c)\right]^2} = t_c^2\,\overline{(\delta\gamma_1)^2} + 2t_c^3\,\overline{\delta\gamma_1\,\delta\gamma_2} + t_c^4\left[\overline{(\delta\gamma_2)^2} + 2\overline{\delta\gamma_1\,\delta\gamma_3}\right] + 2t_c^5\,\overline{\delta\gamma_2\,\delta\gamma_3} + t_c^6\,\overline{(\delta\gamma_3)^2} \qquad (8)$$

where the bar indicates an ensemble average.

It is supposed that $R(t)$ is sampled at short intervals, τ, and that the measurement error has a normal distribution with zero mean and a covariance matrix $\underline{N}$. For convenience we let $t_1 = n\tau$ where n is an integer; then $\underline{N}$ is an n x n matrix. If maximum likelihood estimation is employed, then the covariance matrix of the $\delta\gamma$'s is given by[3]

$$\overline{\underline{\delta\gamma}\;\tilde{\underline{\delta\gamma}}} = \underline{J}^{-1} = \left[\left(\widetilde{\frac{\partial R(t)}{\partial\underline{\gamma}}}\right)\underline{N}^{-1}\left(\frac{\partial R(t)}{\partial\underline{\gamma}}\right)\right]^{-1} \qquad (9)$$

where the usual vector notation is employed, and where, from Eq. (6)

$$\frac{\partial R(t)}{\partial\underline{\gamma}} = \begin{bmatrix} \dfrac{\partial R(t)}{\partial\gamma_1} \\[2ex] \dfrac{\partial R(t)}{\partial\gamma_2} \\[2ex] \dfrac{\partial R(t)}{\partial\gamma_3} \end{bmatrix} = \begin{bmatrix} t \\[2ex] t^2 \\[2ex] t^3 \end{bmatrix} = i\tau \begin{bmatrix} 1 \\[2ex] i\tau \\[2ex] i^2\tau^2 \end{bmatrix} \qquad (10)$$

where i = 1, 2, ... n. $\left(\widetilde{\frac{\partial R}{\partial\underline{\gamma}}}\right)$ is the transpose of $\left(\frac{\partial R}{\partial\underline{\gamma}}\right)$.

For simplicity we assume that the measurement errors on R are uncorrelated from sample to sample, and have the same variance σ^2. (Actually, the variance increases with distance, but for a short

time interval the change of the variance is small.) Then the covariance matrix $\underline{N}$ reduces simply to $\sigma^2 \underline{I}_n$ where $\underline{I}_n$ is the n-dimensional unit matrix, and therefore the general element J_{rs} of the matrix $\underline{J}$ defined in Eq. (9) becomes simply

$$J_{rs} = \frac{1}{\sigma^2} \sum_{i=1}^{n} \left(\frac{\partial R(i\tau)}{\partial \gamma_r}\right)\left(\frac{\partial R(i\tau)}{\partial \gamma_s}\right) \tag{11}$$

The partial derivatives have the form given in Eq. (10). For example

$$J_{12} = \frac{1}{\sigma^2} \sum_{i=1}^{n} (i\tau)(i\tau)^2 \approx \frac{\tau^3 n^4}{4\sigma^2} = \frac{n^4 \tau^4}{4\sigma^2 \tau}$$

$$= \frac{t_1^4}{4\sigma^2 \tau} \tag{12}$$

where we have used the fact that $t_1 = n\tau$ and where the error in the approximation is of order $1/n$. For $\tau = .01$ sec, this error becomes negligible for $t_1 > 1$ sec. The other elements of $\underline{J}$ can be obtained in a similar manner, giving:

$$\underline{J} = \frac{1}{\sigma^2 \tau} \begin{bmatrix} \frac{t_1^3}{3} & \frac{t_1^4}{4} & \frac{t_1^5}{5} \\ \frac{t_1^4}{4} & \frac{t_1^5}{5} & \frac{t_1^6}{6} \\ \frac{t_1^5}{5} & \frac{t_1^6}{6} & \frac{t_1^7}{7} \end{bmatrix} \tag{13}$$

The covariance matrix of the parameter errors is the inverse of this:

$$\underline{J}^{-1} = 15\,\sigma^2\tau \begin{bmatrix} \frac{20}{t_1^3} & -\frac{60}{t_1^4} & \frac{42}{t_1^5} \\ -\frac{60}{t_1^4} & \frac{192}{t_1^5} & -\frac{140}{t_1^6} \\ \frac{42}{t_1^5} & -\frac{140}{t_1^6} & \frac{105}{t_1^7} \end{bmatrix} \tag{14}$$

Substituting the values from Eq. (14) into Eq. (8) gives the result:

$$\left[\overline{\delta R(t_c)}\right]^2 = \frac{15\,\sigma^2\,\tau\,t_c^2}{t_1^3}\left[20 - 120\,\frac{t_c}{t_1} + 270\left(\frac{t_c}{t_1}\right)^2 - 280\left(\frac{t_c}{t_1}\right)^3 + 105\left(\frac{t_c}{t_1}\right)^4\right] \tag{15}$$

When the square root of Eq. (15) is plotted on log-log paper the points for $t_1/t_c < 1/2$ fall on a straight line having a slope of -3.89. For $t_c = 50$ sec, the rms error is therefore given very closely by

$$\frac{\delta R(t_c)_{rms}}{\sigma\,\tau} \approx 9.05 \times 10^6\, t_1^{-3.89} \tag{16}$$

This approximation holds for $\tau << t_1 < 1/2\, t_c$.

Another approximate form for Eq. (15) is

$$\left[\frac{\quad}{\delta\sigma(t_c)}\right]^2 \approx \frac{300\,\sigma^2\,\tau\,t_c^2}{t_1^3}\left[1 - 1.515\,\frac{t_c}{t_1}\right]^4 \tag{17}$$

By comparing coefficients of t_c/t_1 it can be seen that this expression differs from Eq. (15) by less than 1 per cent for any value of t_c/t_1. It has therefore been used to plot the curves shown in Figs. 1 and 5.

Appendix B

PREDICTION OF SLBM IN-PLANE MOTION DURING BOOST WHEN PROPULSION IS KNOWN TO BE CONSTANT THRUST

Suppose that the ground range $R(t)$ of an SLBM is given by a known nonlinear function of a set of parameters $\underline{\alpha} = [\alpha_1, \alpha_2, \cdots \alpha_k]$, i.e.,

$$R(t) = f(\underline{\alpha}, t) \tag{18}$$

If the function $f(\underline{\alpha},t)$ is smoothly varying in $\underline{\alpha}$, then the error $\delta R_c = \delta R(t_c)$ of the predicted value of R at the time of cut-off, t_c, is related to the errors $\delta\alpha_1$ in estimating the α's by the approximate relation:

$$\delta R_c = \frac{\partial f(\underline{\alpha},t_c)}{\partial\alpha_1}\,\delta\alpha_1 + \frac{\partial f(\underline{\alpha},t_c)}{\partial\alpha_2}\,\delta\alpha_2 + \dots + \frac{\partial f(\underline{\alpha},t_c)}{\partial\alpha_k}\,\delta\alpha_k + O(\delta\alpha^2) \tag{19}$$

The higher-order terms can be neglected if either the $\delta\alpha_i$ or the higher partial derivatives are small enough. Whether this is permissible in any particular case obviously depends on $f(\alpha,t)$ and the observation error, and the approximation error should always be checked. For the moment we assume that Eq. (19) is valid. Then the variance of the prediction error is

$$\overline{[\delta R_c]^2} = \left[\frac{\partial f(\underline{\alpha}, t_c)}{\partial \alpha_1}\right]^2 \overline{\delta\alpha_1^2} + \left[\frac{\partial f(\underline{\alpha}, t_c)}{\partial \alpha_2}\right]^2 \overline{\delta\alpha_2^2} + \cdots$$

$$+ 2\left[\frac{\partial f(\underline{\alpha}, t_c)}{\partial \alpha_1}\ \frac{\partial f(\underline{\alpha}, t_c)}{\partial \alpha_2}\right] \overline{\delta\alpha_1\ \delta\alpha_2} + \cdots \tag{20}$$

where $\overline{\delta\alpha_i^2}$ and $\overline{\delta\alpha_i \delta\alpha_j}$ are respectively the variances and covariances of the error in estimating the α's.

It is supposed that the α's are estimated on the basis of measured values of $R(t)$, and that the measurement error is normally distributed with zero mean. It is assumed that $R(t)$ is sampled at short intervals τ, and that at the time of estimation t_1 n samples of $R(t)$ have been obtained; thus $t_1 = n\tau$. Then the joint distribution of all the n measurement errors is normal, with an n-dimensional covariance matrix $\underline{N}$.

If the α's are estimated by use of maximum likelihood, then the estimated value of $\underline{\alpha}$ is given by the solution of the maximum likelihood equations:(3,8)

$$\underline{\tilde{X}N}^{-1} [\underline{R} - \underline{f}(\hat{\alpha})] = \underline{0} \tag{21}$$

$$\text{where } \underset{\sim}{\tilde{X}} = \begin{bmatrix} \frac{\partial f(\underline{\alpha},\tau)}{\partial \alpha_1} & \frac{\partial f(\underline{\alpha},2\tau)}{\partial \alpha_1} & \cdots & \frac{\partial f(\underline{\alpha},n\tau)}{\partial \alpha_1} \\ \frac{\partial f(\underline{\alpha},\tau)}{\partial \alpha_2} & \frac{\partial f(\underline{\alpha},2\tau)}{\partial \alpha_2} & \cdots & \frac{\partial f(\underline{\alpha},n\tau)}{\partial \alpha_2} \\ \vdots & & & \\ \frac{\partial f(\underline{\alpha},\tau)}{\partial \alpha_k} & \cdots & \cdots & \frac{\partial f(\underline{\alpha},n\tau)}{\partial \alpha_k} \end{bmatrix}$$

$\underline{\tilde{X}}$ is the transpose of $\underline{X}$

$$\underline{R} = \begin{bmatrix} R(\tau) \\ R(2\tau) \\ \cdot \\ \cdot \\ R(n\tau) \end{bmatrix} \quad , \quad \underline{f}(\hat{\underline{\alpha}}) = \begin{bmatrix} f(\hat{\underline{\alpha}},\tau) \\ f(\hat{\underline{\alpha}},2\tau) \\ \cdot \\ \cdot \\ f(\hat{\underline{\alpha}},n\tau) \end{bmatrix}$$

$$\text{and } \hat{\underline{\alpha}} = \begin{bmatrix} \hat{\alpha}_1 \\ \cdot \\ \cdot \\ \cdot \\ \hat{\alpha}_k \end{bmatrix} \quad \text{is the estimated value of } \underline{\alpha}.$$

Note that Eq. (21) is a set of k simultaneous equations in $\underline{\alpha}$ that are nonlinear because $\underline{f}(\underline{\alpha})$ is nonlinear. However, here again, if $\underline{f}(\underline{\alpha})$

is smoothly varying it can be expanded in a Taylor series about a nominal value of $\underline{\alpha}$, $\underline{\alpha}_0$, as follows:

$$\underline{f}(\underline{\alpha}) = \underline{f}(\underline{\alpha}_0) + X(\underline{\alpha} - \underline{\alpha}_0) + \cdots \qquad (22)$$

and Eq. (21) can be replaced by a linearized version:

$$\tilde{\underline{X}}\, \underline{N}^{-1} [\underline{R} - \underline{f}(\underline{\alpha}_0) - \underline{X}(\hat{\underline{\alpha}} - \underline{\alpha}_0)] = 0$$

which has the explicit solution:

$$\hat{\underline{\alpha}} - \underline{\alpha}_0 = \left[\tilde{\underline{X}}\, \underline{N}^{-1}\, \underline{X}\right]^{-1} \tilde{\underline{X}}\, \underline{N}^{-1} \left[\underline{R} - \underline{f}(\underline{\alpha}_0)\right] \qquad (23)$$

Note that the matrix $\underline{X}$ of partial derivative is generally not square and it therefore does not have an inverse. Hence the $\underline{X}$'s cannot be cancelled in Eq. (23). However, the matrix product $\left[\tilde{\underline{X}}\, \underline{N}^{-1}\, \underline{X}\right]$ is a k x k symmetric matrix which is generally non-singular and therefore has an inverse.*

Suppose now that $\underline{\alpha}_0$ is actually the correct value of $\underline{\alpha}$. Then $\hat{\underline{\alpha}} - \underline{\alpha}_0$ is the error in the estimation of α. Also $\underline{f}(\underline{\alpha}_0)$ is then the correct value of $\underline{R}$, and $\underline{R} - \underline{f}(\underline{\alpha}_0) = \underline{n}$, the noise. The variance of the estimation error is then

$$\overline{(\hat{\underline{\alpha}} - \underline{\alpha}_0)^2} = \left[\tilde{\underline{X}}\, \underline{N}^{-1}\, \underline{X}\right]^{-1} \tilde{\underline{X}}\underline{N}^{-1}\, \overline{\underline{n}\tilde{\underline{n}}}\, \underline{N}^{-1}\, \underline{X} \left[\tilde{\underline{X}}\underline{N}^{-1}\, \underline{X}\right]^{-1}$$

*If this matrix is singular, this is an indication that the measurements are not sensitive to one or more of the parameters. In this case the ratio of estimation error to measurement error quite properly becomes infinite.

where the symmetry of $\underline{N}^{-1}$ and of $\widetilde{\underline{X}}\underline{N}^{-1}\underline{X}$ permits omission of the transpose symbols at the right. However, $\overline{\underline{n}\widetilde{\underline{n}}} = \underline{N}$, the noise covariance matrix. Hence

$$\overline{(\hat{\underline{\alpha}} - \underline{\alpha}_o)^2} = \left[\widetilde{\underline{X}}\, \underline{N}^{-1}\, \underline{X}\right]^{-1} \equiv \underline{J}^{-1} \tag{24}$$

It is necessary to check the validity of the linear approximation; this can be done by using the rms estimation error computed from Eq. (24) as an indication of the actual error, and then to use this to investigate the magnitude of the higher-order terms of Eqs. (19) and (22). It is then generally possible to place bounds on the magnitude of the noise variance terms for which the approximation is justified. This will be done in the specific application given below.

For simplicity we shall again assume that the measurement errors in R(t) are uncorrelated from sample to sample and have the same variance σ^2. In this case the noise covariance matrix $\underline{N}$ reduces to $\sigma^2 \underline{I}_n$ where $\underline{I}_n$ is the n-dimensional unit matrix. Then the covariance matrix $\underline{J}$ of the estimation error becomes

$$\underline{J} = \frac{1}{\sigma^2} \left[\widetilde{\underline{X}}\, \underline{X}\right]$$

so that the typical element of $\underline{J}$ becomes:

$$J_{rs} = \frac{1}{\sigma^2} \sum_{i=1}^{n} \frac{\partial f(\underline{\alpha}, i\tau)}{\partial \alpha_r} \frac{\partial f(\underline{\alpha}, i\tau)}{\partial \alpha_s} \tag{25}$$

If τ is small, then the summation can be replaced by an integral so that

$$J_{rs} = \frac{1}{\sigma^2 \tau} \int_0^{t_1} \frac{\partial f(\underline{\alpha},t)}{\partial \alpha_r} \frac{\partial f(\underline{\alpha},t)}{\partial \alpha_s} dt \qquad (26)$$

where use has been made of the fact that $\Delta i \rightarrow di = \frac{1}{\tau} d(\tau i) = \frac{1}{\tau} dt$ and that $n\tau = t_1$.

It should be noted that if $f(\underline{\alpha},t)$ is actually a linear function of some of the α's, then neither the covariance matrix of the estimation error nor the variance of the prediction error will be a function of these α's. Thus, suppose that $f(\underline{\alpha},t)$ is a linear function of α_1. Then

$$f(\underline{\alpha},t) = k_1(t)\, \alpha_1 + g(\underline{\alpha},t)$$

where $g(\underline{\alpha},t)$ is not a function of α_1. Then

$$\frac{\partial f(\underline{\alpha},t)}{\partial \alpha_1} = k_1(t)$$

which is not a function of α_1. Thus the matrix $\underline{X}$ defined in Eq. (21) will not be a function of α_1 and neither will the error covariance matrix $\underline{J}^{-1}$ defined in Eq. (24). Again, since the partial derivatives $\partial f(\underline{\alpha},t_c)/\partial \alpha_1$ are not functions of α_1 if $f(\underline{\alpha},t)$ is a linear function of α_1, the prediction error given by Eq. (20) is also independent of these α_1.

EXAMPLE OF A CONSTANT-THRUST TARGET

Consider now an SLBM target on a straight-line constant-thrust trajectory. The range is given by:

$$R(t) = R_o + V_o t + gIt \left[\frac{w_o - rt}{rt} \log \left(\frac{w_o - rt}{w_o} \right) + 1 \right] \qquad (27)$$

where

R_o = range in feet at time t = 0

V_o = velocity in ft/sec at time t = 0

g = acceleration of gravity in ft/sec^2

I = specific impulse of the rocket fuel in sec

w_o = weight of the target vehicle in lb at t = 0

r = rate of fuel expenditure in lb/sec

If we consider the parameters $\alpha_1 = R_o$, $\alpha_2 = V_o$, $\alpha_3 = gI$, and $\alpha_4 = r/w_o$, then

$$f(\underline{\alpha},t) = \alpha_1 + \alpha_2 t + \frac{\alpha_3}{\alpha_4} \left[(1 - \alpha_4 t) \log (1 - \alpha_4 t) + \alpha_4 t \right] \qquad (28)$$

$$\frac{\partial f(\underline{\alpha},t)}{\partial \alpha_1} = 1$$

$$\frac{\partial f(\underline{\alpha},t)}{\partial \alpha_2} = t \qquad (29)$$

$$\frac{\partial f(\underline{\alpha},t)}{\partial \alpha_3} = \frac{1}{\alpha_4} \left[(1 - \alpha_4 t) \log (1 - \alpha_4 t) + \alpha_4 t \right]$$

$$\frac{\partial f(\underline{\alpha},t)}{\partial \alpha_4} = -\frac{\alpha_3}{\alpha_4^2}\left[\log(1-\alpha_4 t) + \alpha_4 t\right] \tag{29}$$

Substituting Eq. (29) into Eq. (26) gives:

$$\left.\begin{aligned}
J_{11} &= \frac{t_1}{\sigma^2\tau} \\
J_{22} &= \frac{t_1^3}{3\sigma^2\tau} \\
J_{33} &= \frac{t_1^3}{\sigma^2\tau u^3}\left[\frac{u}{54}(34u^2 - 21u - 6) - \frac{1}{9}(1+8u)(1-u)^2\log(1-u)\right. \\
&\qquad\qquad \left. - \frac{1}{3}(1-u)^3\log^2(1-u)\right] \\
J_{44} &= \frac{\alpha_3^2 t_1^5}{\sigma^2\tau u^5}\left[\frac{6u - 3u^2 + 2u^3}{6} + (1-u)^2\log(1-u) - (1-u)\log^2(1-u)\right] \\
J_{12} &= J_{21} = \frac{t_1^2}{2\sigma^2\tau} \\
J_{13} &= J_{31} = \frac{\alpha_3 t_1^3}{\sigma^2\tau\, u^2}\left[\frac{3u^2}{4} - \frac{u}{2} - \frac{(1-u)^2}{2}\log(1-u)\right] \\
J_{14} &= J_{41} = \frac{\alpha_3 t_1^3}{\sigma^2\tau\, u^3}\left[(1-u)\log(1-u) + u - \frac{u^2}{2}\right] .
\end{aligned}\right\} \tag{30}$$

$$\left.\begin{aligned}
J_{23} = J_{32} &= \frac{t_1^3}{\sigma^2 \tau u^3}\left[\frac{16u^3 - 3u^2 - 6u}{36} - \frac{1}{6}(1 - 3u^2 + 2u^3)\log(1-u)\right] \\
J_{24} = J_{42} &= \frac{\alpha_3 t_1^4}{\sigma^2 \tau u^4}\left[\frac{1-u^2}{2}\log(1-u) - \frac{1}{12}(4u^3 - 3u^2 - 6u)\right] \\
J_{34} = J_{43} &= \frac{\alpha_3 t_1^4}{\sigma^2 \tau u^4}\left[\frac{1}{2}(1-u)^2\log^2(1-u)\right. \\
&\quad \left. + \frac{1}{6}(1 + 6u - 9u^2 + 2u^3)\log(1-u) + \frac{1}{36}(6u + 21u^2 - 16u^3)\right]
\end{aligned}\right\} \quad (30)$$

where $u = \alpha_4 t_1$

The variances and covariances of the estimated parameters needed in Eq. (20) to determine $\overline{\delta R_c^2}$ are, according to Eq. (24), the elements of the inverse matrix $\underline{J}^{-1}$. The inversion of the 4 x 4 $\underline{J}$ matrix whose elements are given in Eq. (30) is best done by means of a digital computer. However an approximate idea of the solution can be gained by reducing the problem somewhat and approximating the log functions in Eq. (30) by a power series. In this connection it should be noted that if $u_c = \alpha_4 t_c$, $1 - u_c = 1/A$, where A is the mass ratio of the rocket. Hence $0 < u < 1$. For small t_1 the expansion for $\log(1-u)$ therefore can be expected to converge quite well.

The dimensionality of the problem can be reduced by assuming that the initial velocity of the target vehicle is known to be zero.* This

*The computations given below indicate that the major component of the prediction error is contributed by errors in the last two parameters α_3 and α_4. Hence removal of either α_1 or α_2 does not have much effect on the results.

removes the second term from Eq. (27) and eliminates the elements J_{22}, J_{12}, J_{23}, J_{24} from $\underline{J}$. The approximate expressions for the remaining elements of $\underline{J}$, obtained by expanding log(1-u) into a power series about u = 0, become:

$$\left.\begin{aligned}
J_{11} &= \frac{t_1}{\sigma^2_T} \\
J_{33} &= \frac{t_1^3}{\sigma^2_T}\left(\frac{u^2}{20} + \frac{u^3}{36} + \frac{u^4}{63} + \dots\right) \\
J_{44} &= \frac{\alpha_3^2 t_1^5}{\sigma^2_T}\left(\frac{1}{20} + \frac{u}{18} + u^2\left(\frac{13}{7\mathrm{x}36}\right) + \dots\right) \\
J_{13} = J_{31} &= \frac{t_1^2}{\sigma^2_T}\left(\frac{u}{6} + \frac{u^2}{24} + \frac{u^3}{60} + \dots\right) \\
J_{14} = J_{41} &= \frac{\alpha_3 t_1^3}{\sigma^2_T}\left(\frac{1}{6} + \frac{u}{12} + \frac{u^2}{20} + \dots\right) \\
J_{34} = J_{43} &= \frac{\alpha_3 t_1^4}{\sigma^2_T}\left(\frac{u}{20} + \frac{u^2}{24} + \frac{2u^3}{63} + \dots\right)
\end{aligned}\right\} \qquad (31)$$

The inversion of the 3 x 3 $\underline{J}$ matrix is now fairly straightforward. The detailed computations show that the matrix is very nearly singular, i.e., the determinant $|J|$ consists of terms which approximately cancel each other, leaving only a small residue. The computations for the inverse must therefore be fairly exact. This fact must be kept in mind when the larger 4 x 4 matrix is computed by means of the digital

computer. Judging by the results obtained in the 3 x 3 case, the numerical work should be carried to at least ten places to obtain reasonable accuracy. The approximate inverse matrix is:

$$\underline{J}^{-1} = \frac{\sigma^2 \tau}{1+O(u)} \begin{bmatrix} \frac{4(1 + 3u + \ldots)}{t_1} & -\frac{168 + 350.4u + \ldots}{u^2\, t_1^2} & \frac{168 + 290.4u + \ldots}{\alpha_3\, u\, t_1^3} \\ -\frac{168 + 350.4u + \ldots}{u^2\, t_1^2} & \frac{16128 + 20160u + \ldots}{u^4\, t_1^3} & -\frac{16128 + 15120u + \ldots}{\alpha_3\, u^3\, t_1^4} \\ \frac{168 + 290.4u + \ldots}{\alpha_3\, u\, t_1^3} & -\frac{16128 + 15120u + \ldots}{\alpha_3\, u^3\, t_1^4} & \frac{16128 + 10080u + \ldots}{\alpha_3^2\, t_1^5\, u^2} \end{bmatrix} \quad (32)$$

It is clear that the convergence of the elements of the $\underline{J}^{-1}$ matrix is doubtful unless u is very much less than one. The O(u) term appearing in the determinant of $\underline{J}$ has not been computed, but the indications are that the coefficient of the next term is also greater than one, and that, therefore, the O(u) term is not negligible except for very small u.

The mean-square error in the prediction of $R(t_1)$ is given by Eq. (20). Using Eq. (29) for the definition of the partial derivatives, and using only the first term of the elements of $\underline{J}^{-1}$, the expression is:

$$\overline{[\delta R_c]^2} = \frac{\sigma^2\tau}{t_1}\left\{4 + \frac{16128\left[(1-u_c)\log(1-u_c) + u_c\right]^2}{u^6} + \frac{16128\left[\log(1-u_c) + u_c\right]^2}{u^6}\right.$$

$$- \frac{336\left[(1-u_c)\log(1-u_c) + u_c\right]}{u^3} - \frac{336\left[\log(1-u_c) + u_c\right]}{u^3} \tag{33}$$

$$\left. + \frac{32256\left[(1-u_c)\log(1-u_c) + u_c\right]\left[\log(1-u_c) + u_c\right]}{u^6}\right\}$$

where $u_c = \alpha_4 t_c$. Note that α_3 does not appear in this result. Equation (33) has been computed for the following numerical example:

$$t_c = 50 \text{ sec}$$

$$\alpha_4 = .0143 \text{ sec}^{-1}$$

The computation shows that the second, third, and sixth terms of Eq. (33) are very much larger than the other three terms. Thus $\overline{\left[\delta R_c\right]^2}$ decreases with t_1^7; in fact, $\overline{\left[\delta R_c\right]^2}$ can be approximated by

$$\overline{[\delta R]^2} = 16128\,\frac{\sigma^2\tau}{\alpha_4^6 t_1^7}\left[(2 - u_c)\log(1 - u_c) + 2u_c\right]^2 \tag{34}$$

The curves shown in Figs. 2 and 5 were plotted from computer results. The curve for three unknown parameters coincides quite closely with the approximate expression, Eq. (34), for values of $u < .1$; for large values of u the rms error predicted by Eq. (34) is considerably greater than the value obtained by the computer for both the three- and four-parameter cases.

A check on the validity of linear approximations used in this development was undertaken by computing the ratio r of second-order terms to linear terms in Eq. (19) as applied to the constant-thrust propulsion example. It is clear from Eq. (29) that the only non-zero second partials are:

$$\frac{\partial^2 f(\underline{\alpha},t)}{\partial\alpha_3\,\partial\alpha_4} = -\frac{1}{\alpha_4^2}\left[\log(1-\alpha_4 t) + \alpha_4 t\right] \tag{35}$$

and

$$\frac{\partial^2 f(\underline{\alpha},t)}{\partial\alpha_4^2} = \frac{\alpha_3}{\alpha_4^3}\;\frac{2(1-\alpha_4 t)\log(1-\alpha_4 t) + 2\alpha_4 t - \alpha_4^2 t^2}{1-\alpha_4 t} \tag{36}$$

Hence the ratio r is given by:

$$r = \frac{\dfrac{\partial^2 f}{\partial\alpha_3\,\partial\alpha_4}\,\delta\alpha_3\,\delta\alpha_4 + \dfrac{1}{2}\,\dfrac{\partial^2 f}{\partial\alpha_4^2}\,(\delta\alpha_4)^2}{\displaystyle\sum_{i=1}^{4}\frac{\partial f}{\partial\alpha_i}\,\delta\alpha_i} \tag{37}$$

where f is shorthand for $f(\underline{\alpha},t)$. The ratio is evaluated by substituting the rms values of the $\delta\alpha_i$ in this expression, and if r is less than some arbitrary figure (say, .1) the linear approximation may be taken to be reasonably accurate.

This computation has been performed for the following values of the parameters:

$$t_c = 50 \text{ sec}$$

$$\alpha_4 = .0143 \text{ sec}^{-1}$$

$$\alpha_3 = 1970 \text{ ft/sec}$$

corresponding to the typical trajectory used throughout this Memorandum and given in Appendix E. It should be noted that the contribution of the second-order terms is no longer independent of α_3 and a value must therefore be specified. A curve of $\sigma\sqrt{\tau}$ vs t_1 for $r = .1$ is given in Fig. 7. From this curve it can be seen that if, for instance, $\sigma\sqrt{\tau} = 100$ ft sec$^{1/2}$ then the second-order terms in Eq. (19) are less than 10 per cent of the first-order terms for $t_1 > 18$ sec. The curve rises rather rapidly with t_1 so that for the same value of $\sigma\sqrt{\tau}$ the approximation error is less than 1 per cent for $t_1 > 28$ sec.

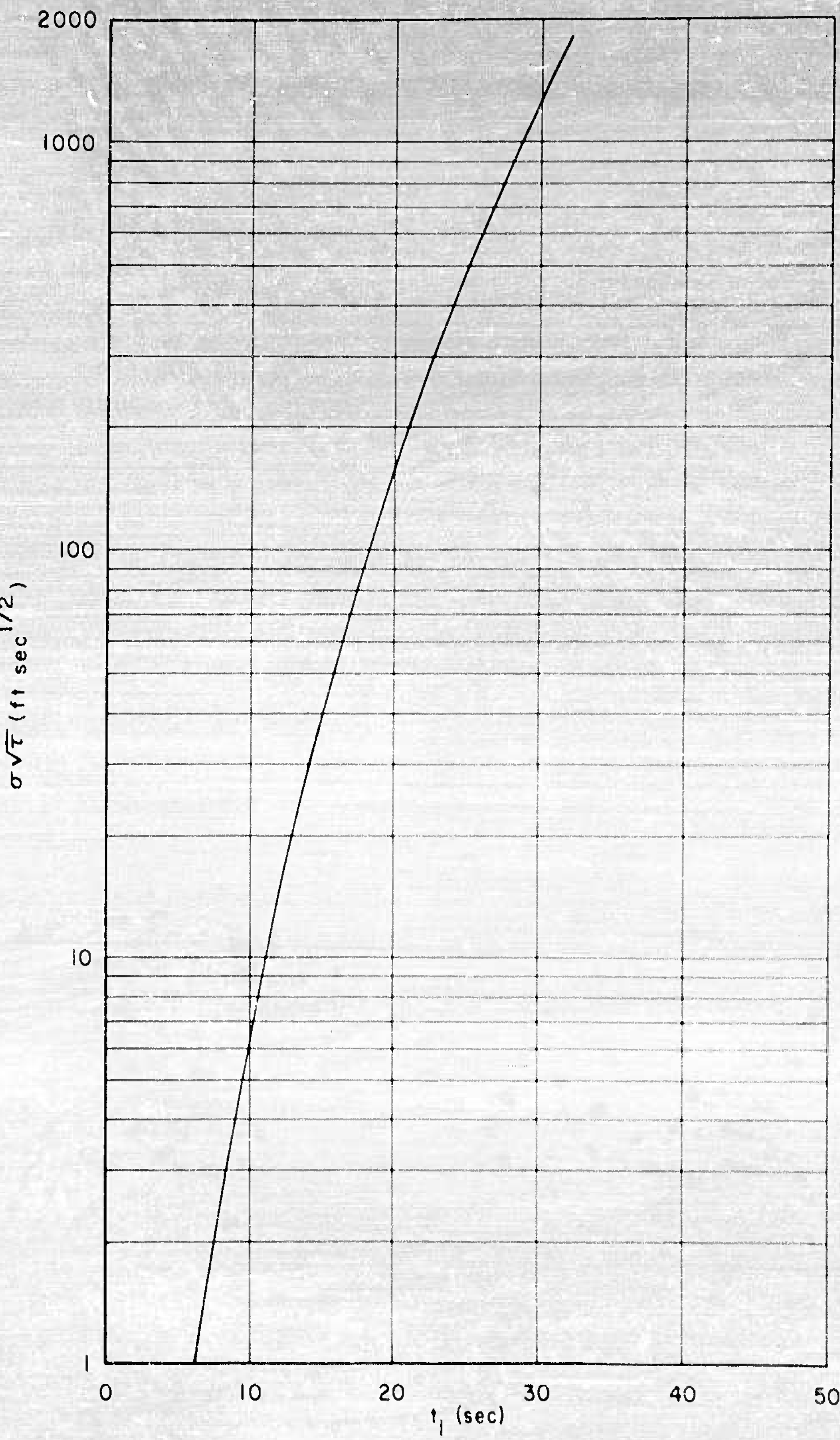

Fig. 7 — Values of $\sigma\sqrt{\tau}$ versus t_1 for which ratio of second-order terms to first-order terms in Eq. (19) is 0.1

Appendix C

PREDICTION OF SLBM IN-PLANE POSITION DURING BOOST WHEN PROPULSION IS KNOWN TO BE CONSTANT-THRUST AND WHEN ONLY THE TIME ORIGIN IS UNKNOWN

It is supposed that the ground range of the target missile is given by

$$R = R(t - s) \tag{38}$$

where s is unknown, but where the functional form of R is known exactly. Discrete observations are made on R starting at some time $t = 0$ and ending at time t_1. The time of the first observation t_o occurs some seconds after the SLBM is launched. At time t_1 it is desired to estimate $R(t_c) = R_c$; in general $t_c > t_1$. The observations on R are spaced τ seconds apart and are corrupted by additive noise, which is assumed to be white, gaussian, and stationary, with zero mean and variance σ^2.

The problem of estimating s is essentially the problem considered by P. M. Woodward.(9,10) Woodward shows that the a posteriori probability of s, given the observed values of R, is given by:

$$p(s/R) = k\ p(s)\ e^{g(s)} \tag{39}$$

where k is a normalizing constant chosen such that

$$\int_{-\infty}^{\infty} p(s/R)\ ds = 1 \tag{40}$$

p(s) is the a priori probability density of s which is assumed to be uniform in the sequel. Under the assumption of white, gaussian,

stationary noise with zero mean and variance σ^2

$$g(s) = \frac{1}{\sigma^2} \sum_{i=1}^{n} \left[-\frac{1}{2} R^2(t_i - s) + R(t_i - s)\, R(t_i - s_o) + R(t_i - s)\, n(t_i) \right] \tag{41}$$

where s_o is the true value of s (unknown to the observer), $n(t_i)$ is the noise, and $t_i = i\tau$ is the i^{th} observation instant. The upper limit of summation n is such that $t_n = n\tau = t_1$, the time at which the observations cease and the prediction is made.

For s near s_o, $g(s)$ can be expanded in a Taylor series:

$$g(s) = g(s_o) + (s - s_o) \left.\frac{\partial g}{\partial s}\right|_{s=s_o} + \frac{1}{2}(s - s_o)^2 \left.\frac{\partial^2 g}{\partial s^2}\right|_{s=s_o} + \ldots \tag{42}$$

From Eq. (41) it is clear that

$$g(s_o) = \frac{1}{2\sigma^2} \sum_{i=1}^{n} \left[R^2(t_i - s_o) + 2R(t_i - s_o)\, n(t_i) \right] \tag{43}$$

$$\left.\frac{\partial g}{\partial s}\right|_{s_o} = g'(s_o) = -\frac{1}{\sigma^2} \sum_{i=1}^{n} n(t_i)\, R'(t_i - s_o) \tag{44}$$

$$\left.\frac{\partial^2 g}{\partial s^2}\right|_{s_o} = g''(s_o) = -\frac{1}{\sigma^2} \sum_{i=1}^{n} \left\{ \left[R'(t_i - s_o) \right]^2 + n(t_i)\, R''(t_i - s_o) \right\} \tag{45}$$

Woodward[10] has shown that if $\frac{1}{2\sigma^2}\sum_{i=1}^{n} R^2(t_i - s_o) >> 1$ and if

$\frac{1}{\sigma^2}\sum_{i=1}^{n}\left[R'(t_i - s_o)\right]^2 >> 1$, then the a posteriori probability $p(s/R)$ can be obtained approximately by ignoring the terms of Eqs. (43), (44) and (45) containing $n(t)$. In this case, the $g'(s_o)$ term is neglected. Also, $g(s_o)$ is not a function of s and can be combined with the normalizing constant of Eq. (39). The expression for the approximate a posteriori probability becomes:

$$p(s/R) \approx k' \, e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left[R'(t_i - s_o)\right]^2 (s - s_o)^2} \tag{46}$$

where k' is a new normalizing constant containing $e^{g(s_o)}$ and also the a priori probability $p(s)$ which was assumed to be uniform (i.e., constant). Equation (46) shows the a posteriori probability to be approximately gaussian with mean s_o and variance

$$\overline{(s - s_o)} = \overline{\delta s^2} = \frac{\sigma^2}{\sum_{i=1}^{n}\left[R'(t_i - s_o)\right]^2} \tag{47}$$

It might be noted that this variance is identical with the one that would have been obtained by straightforward application of maximum-likelihood estimation of s_o.

The predicted error at time $t = t_c$ is now obtained by direct application of methods used in Appendix B. Since only a single parameter

is assumed to be uncertain, the prediction error is given by

$$\overline{\delta R_c^2} = \left[\frac{\partial R(t - s)}{\partial s}\right]^2_{\substack{s=s_o \\ t=t_c}} \overline{\delta s^2} \tag{48}$$

It is again convenient to convert the summation into an integral. Also, it is convenient to let $t_o = 0$. Then

$$\sum_{i=1}^{n} \left[R'(t_i - s_o)\right]^2 \rightarrow \frac{1}{\tau} \int_0^{t_1} \left[R'(t - s_o)\right]^2 dt \tag{49}$$

and

$$\overline{\delta R_c^2} = \left[\frac{\partial R(t-s)}{\partial s}\right]_{\substack{s=s_o \\ t=t_c}} \frac{\sigma^2 \tau}{\int_0^{t_1} \left[R'(t-s_o)\right]^2 dt} \tag{50}$$

SPECIFIC EXAMPLE

As in Appendix B, we now assume that the form of R is given by

$$R(t-s) = \alpha_1 + \alpha_2(t-s) + \frac{\alpha_3}{\alpha_4}\left\{\left[1-\alpha_4(t-s)\right] \log\left[1-\alpha_4(t-s)\right] + \alpha_4(t-s)\right\} \tag{51}$$

where the α's are defined as in Eq. (28), Appendix B. Then

$$R'(t-s) = \frac{dR}{ds} = -\alpha_2 + \alpha_3 \log\left[1-\alpha_4(t-s)\right] \tag{52}$$

Hence

$$\frac{\overline{\delta R_c^2}}{\sigma^2 \tau} = \frac{\alpha_2^2 - 2\alpha_2\alpha_3 \log(1-u_c) + \alpha_3^2 \log^2(1-u_c)}{\alpha_2^2 t_1 - \frac{2\alpha_3}{\alpha_4}(\alpha_3+\alpha_2)\left[(1+u_o)\log(1+u_o) - (1-u)\log(1-u) + u_o - u\right] + \frac{\alpha_3^2}{\alpha_4}\left[(1+u_o)\log^2(1+u_o) - (1-u)\log^2(1-u)\right]} \tag{53}$$

where $u_o = \alpha_4 s_o$

$u_c = \alpha_4(t_c - s_o)$

$u = \alpha_4(t_1 - s_o)$

For small values of u and u_o it is convenient to replace $\log(1+u_o)$ and $\log(1-u)$ by a series. A particularly simple expression results when $\alpha_2 = 0$:

$$\frac{\overline{\delta R_c^2}}{\sigma^2 \tau} = \frac{\alpha_4 \log^2(1 - u_c)}{\frac{1}{3}(u^3+u_o^3) + \frac{1}{4}(u^4-u_o^4) + \frac{11}{60}(u^5+u_o^5) + \ldots} \tag{54}$$

From this it appears that for small t_1 $\overline{\delta R_c^2}$ decreases as $u^3 + u_o^3$.

In general,

$$\frac{\overline{\delta R_c^2}}{\sigma^2 \tau} = \frac{\alpha_4\left[k - \log(1 - u_c)\right]^2}{\alpha_4 k^2 t_1 + k(u^2 - u_o^2 + \frac{u^3+u_o^3}{3} + \frac{u^4-u_o^4}{6} + \ldots) + \frac{u^3+u_o^3}{3} + \frac{u^4-u_o^4}{4} + \frac{11(u^5-u_o^5)}{60} + \ldots} \tag{55}$$

where $k = \alpha_2/\alpha_3$. For positive k this result is usually smaller than for $k = 0$. Hence curves are plotted only for $\alpha_2 = 0$. These are shown in Fig. 4.

Appendix D

PREDICTION OF SLBM IN-PLANE MOTION DURING BOOST WHEN PROPULSION IS KNOWN TO BE CONSTANT-THRUST AND WHEN ONLY TIME ORIGIN AND INITIAL POSITION ARE UNKNOWN

The results of Appendix C assume complete knowledge of all the parameters. It is more realistic to assume that the initial position parameter, α_1, is only partially known -- that is, one does not know in advance when the SLBM was launched nor its position at launch. In this section it is therefore assumed that α_2, α_3, and α_4 are known exactly; that the a priori probability density of α_1 is gaussian with a mean α_m and variance σ_α^2; and that s, the time origin, has a uniform distribution as in Appendix C. (It can be easily demonstrated that the exact form of the a priori probability distribution of α_1 or of s is not important after the first few seconds of observation time.)

Under these conditions, the ground range of the target missile can be expressed in the form:

$$R(t) = \alpha_1 + f(t - s)$$

where f(t - s) is an exactly known function. The a posteriori probability of α_1 and s, given the observed values of R, is then given by

$$p(s, \alpha_1|R) = k\, p(s, \alpha_1)\, e^{g(s,\, \alpha_1)} \tag{56}$$

where k is again a normalizing constant, and where

$$g(s, \alpha_1) = -\frac{1}{2\sigma^2} \sum_{i=1}^{n} \left[R_i - \alpha_1 - f(t_i - s) \right]^2 \tag{57}$$

The noise is white, gaussian, with variance σ^2, as in Appendix C, and the R_i are the observed values of $R(t)$. If the noise is $n(t)$ and if the true values of α_1 and s are α_{10} and s_o respectively, then the observed value of R is

$$R(t_i) = R_i = \alpha_{10} + f(t_i - s_o) + n(t_i)$$

and therefore

$$g(s,\alpha_1) = -\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left[\alpha_{10} + f(t_i - s_o) + n(t_i) - \alpha_1 - f(t_i - s)\right]^2 \qquad (58)$$

As in Appendix C we expand $g(s,\alpha_1)$ about s_o and α_{10}, and we omit terms containing $\overline{n}(t)$ in accordance with Woodward's argument.(10) Under these conditions it is easily shown that

$$g(s_o,\alpha_{10}) = 0 \qquad (59)$$

$$\left.\frac{\partial g}{\partial\alpha_1}\right|_{\substack{s=s_o\\ \alpha_1=\alpha_{10}}} = \left.\frac{\partial g}{\partial s}\right|_{\substack{s=s_o\\ \alpha_1=\alpha_{10}}} = 0 \qquad (60)$$

$$\left.\frac{\partial^2 g}{\partial\alpha_1^2}\right|_{\substack{s=s_o\\ \alpha_1=\alpha_{10}}} = -\frac{n}{\sigma^2} = -\frac{t_1}{\sigma^2\tau} \quad \text{since } n = t_1/\tau \qquad (61)$$

$$\left.\frac{\partial^2 g}{\partial s^2}\right|_{\substack{s=s_o\\ \alpha_1=\alpha_{10}}} = -\frac{1}{\sigma^2}\sum_{i=1}^{n}\left[f'(t_i - s_o)\right]^2 \qquad (62)$$

$$\left.\frac{\partial^2 g}{\partial s \partial \alpha_1}\right|_{\substack{s=s_o \\ \alpha_1=\alpha_{10}}} = \frac{1}{\sigma^2} \sum_{i=1}^{n} f'(t_i - s_o) \tag{63}$$

where $f'(t_i - s_o) = \left.\frac{\partial f(t_i - s)}{\partial s}\right|_{s=s_o}$

Hence

$$g(s,\alpha_1) \approx - \frac{1}{2\sigma^2} \left[\frac{t_1}{\tau} (\alpha_1 - \alpha_{10})^2 - 2 \sum_{i=1}^{n} f'(t_i - s_o)(\alpha_1 - \alpha_{10})(s - s_o) \right.$$

$$\left. + \sum_{i=1}^{n} \left[f'(t_i - s_o) \right]^2 (s - s_o)^2 \right] \tag{64}$$

Also in view of the assumptions made about the a priori probability distributions of α_1 and s, we have

$$p(s,\alpha_1) = k_1 e^{-\frac{1}{2\sigma_\alpha^2}(\alpha_1 - \alpha_m)^2} \tag{65}$$

By means of some algebra the a posteriori distribution $p(s,\alpha_1|R)$ can then be put in the form

$$p(s,\alpha_1|R) = k_2 e^{g_1(s,\alpha_1)} \tag{66}$$

where

$$g_1(s,\alpha_1) = -\frac{1}{2\sigma^2}\left\{\frac{t_1}{\tau}\left(1+\frac{\sigma^2\tau}{\sigma_\alpha^2 t_1}\right)\left(\alpha_1 - \frac{\alpha_{10}\,\sigma_\alpha^2 t_1 + \alpha_m \sigma^2 \tau}{\sigma_\alpha^2 t_1 + \sigma^2\tau}\right)^2\right.$$

$$\left. -2\sum_{i=1}^{n} f'(t_i - s_o)(\alpha_1 - \alpha_{10})(s - s_o) + \sum_{i=1}^{n}\left[f'(t_i - s_o)\right]^2 (s-s_o)^2\right\} \quad (67)$$

The mean value of α_1 in this expression is a linear combination of the a priori mean value α_m and the true value α_{10}. The a priori mean value would be different from the true value only if the measurement of α_1 that results in the a priori distribution were biased. There is no particular reason for assuming the existence of such a bias here, and we therefore assume that $\alpha_m = \alpha_{10}$. It might be noted that if there were a bias in the measurement, its effect would in any case become negligible if $\sigma_\alpha^2 t_1 >> \sigma^2\tau$. Unless σ_α^2 is very much smaller than σ^2--in which case one would be justified in considering α_1 to be known exactly--the effect of the bias is therefore negligible after only a few seconds of observation time.

It is also evident that if $\sigma_\alpha \approx \sigma$

$$1 + \frac{\sigma^2\tau}{\sigma_\alpha^2 t_1} \approx 1 \text{ for } t_1 >> \tau \quad (68)$$

Hence, if τ is on the order of .01 sec or less, after about 1 sec $g_1(s,\alpha_1)$ is given to a very good approximation by

$$g_1(s,\alpha_1) = -\frac{1}{2\sigma^2}\left\{\frac{t_1}{\tau}(\alpha_1-\alpha_{10})^2 - 2\sum_{i=1}^{n} f'(t_i-s_o)(\alpha_1-\alpha_{10})(s-s_o)\right.$$

$$\left. + \sum_{i=1}^{n}\left[f'(t_i-s_o)\right]^2 (s-s_o)^2\right\} \qquad (69)$$

$$= g(s,\alpha_1)$$

i.e., the a posteriori distribution of s and α_1 becomes completely independent of a priori knowledge of α_1. Since Eq. (69) also applies to the case of no a priori information about α_1, which represents a least favorable situation, this expression will be used in all subsequent computations.

The variances and covariances of α_1 and s can now be obtained directly from Eq. (69). Note that this equation is in the form:

$$g_1(s,\alpha_1) = -\frac{1}{2}\left[A(\alpha_1-\alpha_{10})^2 - 2B(\alpha-\alpha_{10})(s-s_o) + C(s-s_o)^2\right]$$

Hence

$$\overline{(\alpha_1 - \alpha_{10})^2} = \frac{C}{AC - B^2}$$

$$\overline{(s - s_o)^2} = \frac{A}{AC - B^2} \qquad (70)$$

$$\overline{(\alpha_1 - \alpha_{10})(s - s_o)} = \frac{B}{AC - B^2}$$

As in Appendix C the prediction error at time t_c is obtained from

$$\overline{\left[\delta R_c\right]^2} = \left[\frac{\partial R_c}{\partial\alpha_1}\right]^2 \overline{(\alpha_1-\alpha_{10})^2} + \left[\frac{\partial R_c}{\partial s}\right]^2 \overline{(s-s_o)^2}$$
$$+ 2\left[\frac{\partial R_c}{\partial\alpha_1}\right]\left[\frac{\partial R_c}{\partial s}\right]\overline{(\alpha_1-\alpha_{10})(s-s_o)} \tag{71}$$

where the partial derivatives are all evaluated for $s = s_o$, $\alpha = \alpha_{10}$, $t = t_c$. If $R(t - s)$ is given by Eq. (51), Appendix C, then

$$\frac{\partial R_c}{\partial\alpha_1} = 1$$
$$\left.\frac{\partial R_c}{\partial s}\right|_{s=s_o} = -\alpha_2+\alpha_3 \log(1-u_c) \tag{72}$$

where $u_c = \alpha_4(t_c - s_o)$.

The two summations appearing in Eq. (69) can be approximated by integrals, as usual. The final result for $\overline{[\delta R_c]^2}$ is then:

$$\frac{\overline{[\delta R_c]^2}}{\sigma^2\tau} = \frac{\alpha_2 t_1 - 2\frac{\alpha_3}{\alpha_4}(\alpha_3+\alpha_2)[g(-u_o)-g(u)] + \frac{\alpha_3^2}{\alpha_4}[h(-u_o)-h(u)] + 2\left[\alpha_2-\alpha_3\log(1-u_c)\right]\left\{\alpha_2 t_1-\frac{\alpha_3}{\alpha_4}\left[g(-u_o)-g(u)\right]\right\} + \left[\alpha_2-\alpha_3\log(1-u_c)\right]^2 t_1}{\frac{\alpha_3^2}{\alpha_4^2}\left\{(u+u_o)[h(-u_o)-h(u)] - 2(u+u_o)\left[g(-u_o)-g(u)\right] - \left[g(-u_o)-g(u)\right]^2\right\}} \tag{73}$$

In this expression $u = \alpha_4(t_1 - s_o)$

$$u_o = \alpha_4 \, s_o$$

$$g(u) = (1 - u) \log(1 - u) + u \tag{74}$$

$$h(u) = (1 - u) \log^2(1 - u)$$

This expression is again considerably simplified if $\alpha_2 = 0$, and this is the only case for which detailed computations have been performed. In this case $\overline{[\delta R_c]^2}$ is independent of α_3.

The results for this case calculated by a digital computer from Eq. (73) for several values of s_o are presented in Fig. 3.

Appendix E

TYPICAL TRAJECTORIES

In all the computations performed in this Memorandum, a typical SLBM trajectory has been assumed when specific parameter values were required. The ground-range function corresponding to this trajectory is shown in Fig. 8. The equation of this function is

$$R(t) = \frac{1970}{.0143}\left[(1 - .0143\ t)\ \log(1 - .0143\ t) + .0143\ t\right] \tag{75}$$

Figure 8 also presents an approximation to Eq. (75) by means of a third-order polynomial function. The coefficients of this polynomial have been chosen so that the polynomial coincides with Eq. (75) for t = 0, 20, 40, and 60 sec. The equation for the polynomial approximation is:

$$R(t) = 120.66\ t + 4.75\ t^2 + .2558\ t^3 \tag{76}$$

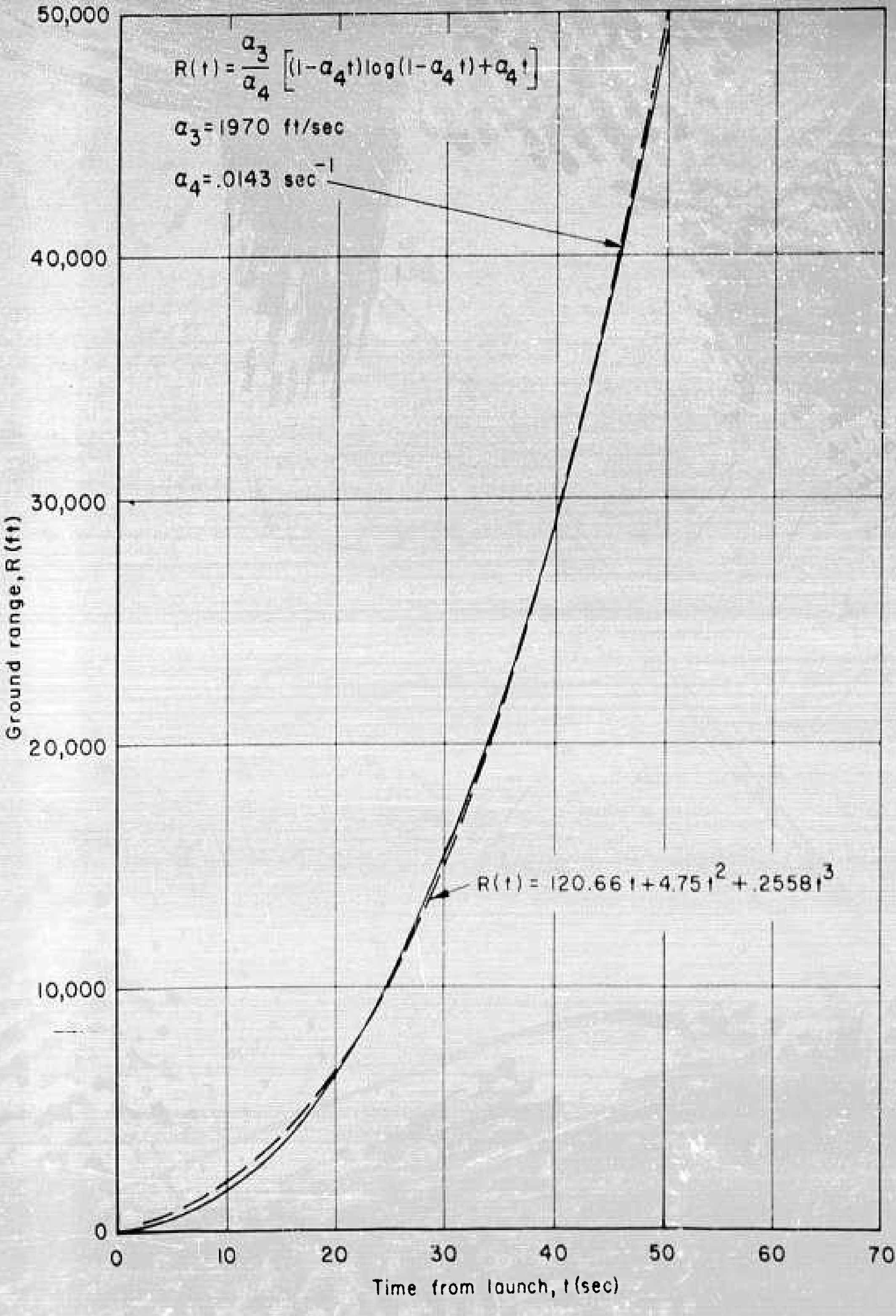

Fig. 8 — Range functions for typical trajectory

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